

# ENTRAINMENT CONTROL OF CHAOS NEAR UNSTABLE PERIODIC ORBITS

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**Abstract.** It is demonstrated that improved entrainment control of chaotic systems can maintain periodic goal dynamics near unstable periodic orbits without feedback. The method is based on the optimization of goal trajectories and leads to small open-loop control forces.

## 1. Introduction

In recent years, a large number of investigations has been made into the control of chaotic dynamics, and many techniques have been applied in simulations and experiments (see, for instance, [1] for an overview).

Two techniques have been proposed for an *open-loop* control of chaos. The first approach is related to vibrational methods, as a scalar periodic perturbation is applied to the chaotic system. Usually, the control signals are sinusoidal [2] or two-mode forces [3]. Recently it has been shown that the method can be improved by use of optimized multimode signals which are more complex [4].

The second method uses equations of motion and a specific goal dynamics to derive vector control forces. If the goal trajectory is suitably chosen, the chaotic system under control converges to the goal. Therefore, this method is often referred to as entrainment control [5]. In this paper it is shown how entrainment control can be improved with respect to small control forces. For this purpose, a specific property of chaotic systems is exploited: dense unstable periodic orbits (UPOs).

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In fact, UPOs are common goal orbits of most feedback techniques employed for control of chaos (see, e.g., [6]). Because a UPO is a natural, but unstable motion of the system, control forces have to be applied only for transfer to and stabilization of the orbit. In the ideal (noiseless) case, the stabilization forces tend to zero once the UPO is actually reached. Therefore, feedback control of UPOs can be maintained with very small forces in low noise systems. Despite the power of such methods, however, there are situations where one wants to or has to dispense with a feedback from the system. Then, open-loop techniques are needed.

While there exists a theory for open-loop stabilization of unstable fixed points (vibrational control, see [7]), a counterpart for stabilization of unstable periodic orbits is still lacking to the author's knowledge. Although it is supposed that the scalar periodic perturbation methods mentioned above usually stabilize periodic dynamics in the vicinity of UPOs, this has not been shown yet explicitly. The underlying mechanism, possibly some resonance phenomenon, as well as the exact final dynamics are still unknown. This is different for entrainment control, where the appropriate dynamics is given *a priori*, and which is described in the following.

## 2. Entrainment control

We start with a nonlinear dynamical system in the continuous time domain, which is influenced by a control signal vector. The equations of motion are supposed to be known, according to the ordinary differential equation

$$\frac{d\mathbf{x}(t)}{dt} = \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \quad (1)$$

with the system state  $\mathbf{x}$  in the state space  $\mathbb{R}^n$ , the control vector  $\mathbf{u}$  in the control signal space  $\mathbb{R}^s$  and the vector field  $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}^n$ . Let  $\mathbf{z}$  be a goal dynamics generated by a vector field  $\mathbf{g}$ ,  $\dot{\mathbf{z}}(t) = \mathbf{g}(\mathbf{z}(t))$ ,  $\mathbf{z}(t_0) = \mathbf{z}_0$ . To introduce the goal dynamics as solutions of Eq. (1), we have to apply forces  $\mathbf{u}(t)$  which solve the equation

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{g}(\mathbf{x}(t)). \quad (2)$$

Thus, the vector field  $\mathbf{f}$  is simply changed to  $\mathbf{g}$  by the control forces. To do this, however, feedback from the system is necessary, as the actual system state  $\mathbf{x}(t)$  appears in Eq. (2). The main point of the entrainment control method is to eliminate  $\mathbf{x}(t)$  by the assumption that the system is already located in the initial goal state  $\mathbf{z}_0$  when the control is started at  $t_0$ . If so, the correct control signal  $\mathbf{u}(t)$  is given by the solution of the equation

$$\mathbf{f}(\mathbf{z}(t), \mathbf{u}(t)) = \mathbf{g}(\mathbf{z}(t)) = \dot{\mathbf{z}}(t). \quad (3)$$

Equation (3) can be solved without any system state measurement; in fact, not even a generating vector field  $\mathbf{g}$  has to be given: it is sufficient to know the goal trajectory itself and its time derivative (velocity) for the control time interval,  $\{\mathbf{z}(t), \dot{\mathbf{z}}(t)\}_{t \in [t_0, t_1]}$ .

There are several conditions that have to be fulfilled to make the entrainment control scheme work. First of all, Eq. (3) has to be solvable for  $\mathbf{u}(t)$ . This is trivially true for simple vector additive forces:

$$\mathbf{f}(\mathbf{z}(t), \mathbf{u}(t)) = \tilde{\mathbf{f}}(\mathbf{z}(t)) + \mathbf{u}(t) \Rightarrow \mathbf{u}(t) = \dot{\mathbf{z}}(t) - \tilde{\mathbf{f}}(\mathbf{z}(t)) \quad (4)$$

We restrict our discussion to such forces, which are the most common in the literature on entrainment control. However, problems immediately arise if, e.g., the control space dimension  $s$  is less than the state space dimension  $n$  (a treatment of general control influence with  $s=d$  can be found in [8]).

The next condition to be satisfied is asymptotic stability of the goal trajectory. While control forces according to Eq. (3) ensure that the goal trajectory is a solution of the controlled system Eq. (1), there is no statement about whether nearby located system states are attracted by it – in other words, whether entrainment occurs in a vicinity of  $\mathbf{z}(t)$  according to  $\lim_{t \rightarrow \infty} |\mathbf{x}(t) - \mathbf{z}(t)| = 0$ . Even if this is the case, one needs a large basin of attraction (ideally the whole phase space) to make the method work for a large set of possible initial states distant from  $\mathbf{z}(t)$ . Statements about basins are very hard to find (compare [9]), but a discussion of stability can more easily be made. For simple vector additive control, Eq. (4), we call all points in phase space where all eigenvalues of the Jacobian of  $\mathbf{f}$  have negative real part, *convergent regions* (see also [8, 9]). Goal trajectories entirely located in convergent regions turn out to be asymptotically stable, if their time derivative  $\dot{\mathbf{z}}(t)$  is sufficiently bounded.

### 3. Optimization of the goal trajectory

Because of the stated stability aspects, goal trajectories are usually chosen to be located in convergent regions. This results in a typical drawback of the method: Due to very little overlap of convergent regions and unperturbed chaotic attractor, such a goal dynamics is quite different to the natural system dynamics. Consequently, the system is strongly altered by control, and control forces are large. In fact, they have to be of about the magnitude of the velocities appearing in the uncontrolled system in order to pull the movement into convergent regions.

To attack this problem, one has to realize that location in convergent regions is not a necessary condition for stability of a goal trajectory. The resulting dynamics of chaotic systems controlled by periodic perturbation methods indeed suggest that stable dynamics can also be achieved in the

chaotic attractor region, especially near a UPO. Control forces are smaller then, as the natural dynamics is only slightly altered. An extreme case would be to consider a UPO itself as a goal for entrainment control: we get zero control forces according to Eq. (3). However, such a goal is of course unstable. It has to be at least slightly changed to result in a stable one. To this end, a family of deformations of a UPO is considered in the following.

Let  $\mathbf{z}_{UPO}(t)$  denote a known UPO of the chaotic system with period  $T=2\pi/\omega$ . We chose a finite Fourier series as a deformation  $\Delta\mathbf{z}(t)$ , and also include a linear time transformation by a factor  $\eta$ . The family of goal trajectories now reads

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{z}_{UPO}(\eta t) + \Delta\mathbf{z}(\eta t) \\ &= \mathbf{z}_{UPO}(\eta t) + \sum_{m=0}^M (\mathbf{a}^m \cos(m\omega\eta t) + \mathbf{b}^m \sin(m\omega\eta t)).\end{aligned}\quad (5)$$

It is parametrized by  $\eta \in \mathbb{R}$  and the Fourier coefficient vectors  $\mathbf{a}^m, \mathbf{b}^m \in \mathbb{R}^n$ . Since  $\mathbf{b}^0$  has no effect on  $\mathbf{z}(t)$ , a deformation (or a goal) is characterized by a total of  $d = (2M+1)n + 1$  real numbers.

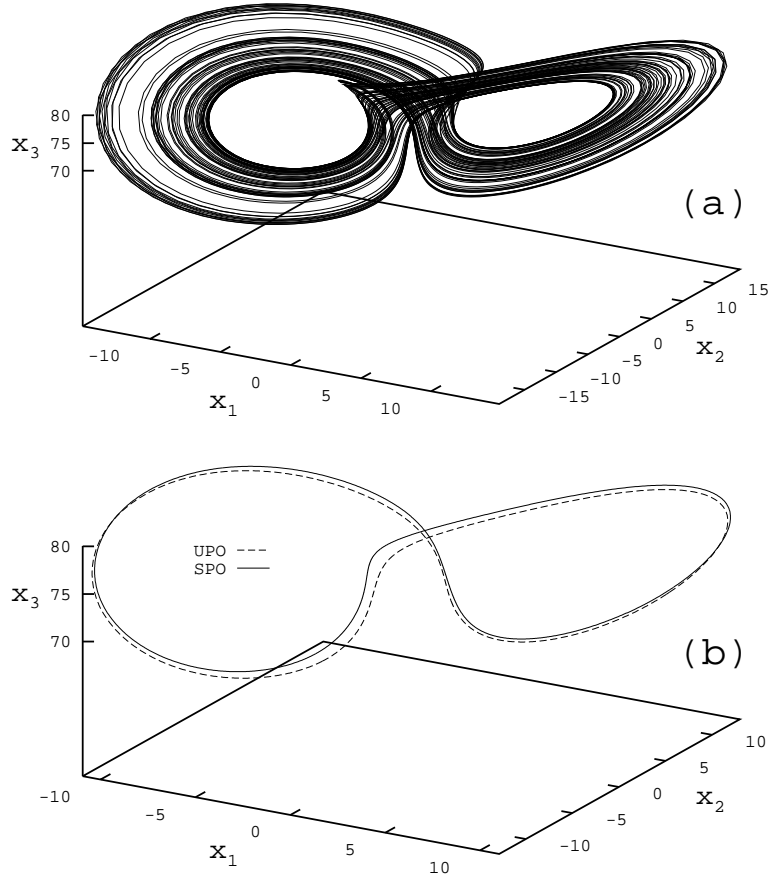
Now, we formulate the determination of advantageous deformation parameters that lead to a stable goal trajectory with small control forces as an optimization problem. A real number according to a cost function is assigned to each probed set of parameters. The cost assesses the stability of the chosen goal trajectory (which is determined numerically) as well as the magnitude of the resulting control forces:

$$cost = \begin{cases} \|\mathbf{u}(t)\|_{max} + \gamma[\exp(\mu) - 1] & : \mu < 1, \\ \|\mathbf{u}(t)\|_{max} + \gamma[\exp(\mu) - 1] + C & : \mu \geq 1 \end{cases}\quad (6)$$

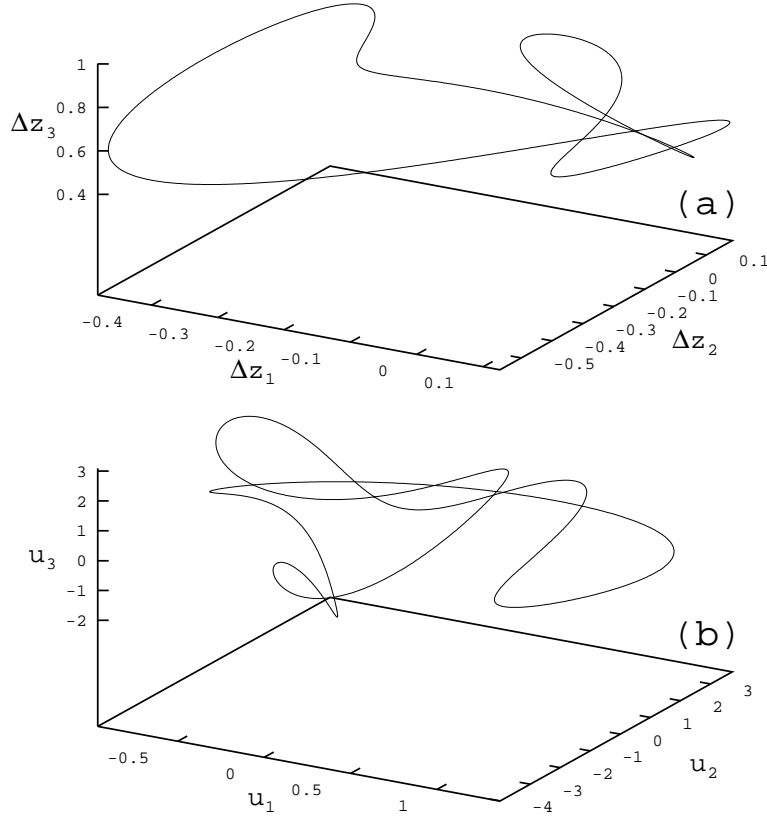
Here,  $\mu = \max_i\{|\mu_i|\}$  is the maximum absolute value of the characteristic (Floquet) multipliers of the goal orbit. These are well defined, as the goal is periodic, and they are calculated by integration of the variational equations [10]. Instability is indicated by  $\mu > 1$  and causes high cost via a large positive penalty term  $C$ . The cost function further includes an  $\exp(\mu)$  and the maximum norm of the resulting forces. The weight of stability with respect to magnitude of forces can be adjusted by  $\gamma$ . The global minimum of the cost function in the deformation parameter space corresponds to the best goal trajectory in sense of the chosen balance between stability and small forces.

For various reasons, a direct analytic treatment of the given optimization problem is usually not possible: the UPO is not known in analytic form, a direct expression for stability of a deformed UPO is missing (the variational equations have to be integrated), and the cost function is not continuous. Consequently, numerical methods are employed. The UPO is

represented by a periodic cubic spline interpolation, and  $\mu$  is calculated via numerical integration of the variational equations. The optimization is done by a numerical technique that can handle high-dimensional problems with rough and rapidly varying cost functions. For this purpose, the stochastically guided algorithm `amebsa` from [11] was chosen; this is a combination of simulated annealing and the downhill simplex method.



*Figure 1.* (a): Chaotic attractor of the Lorenz system ( $\sigma = 10.0$ ,  $r = 75.0$ ,  $b = 0.4$ ). (b): Embedded unstable periodic orbit (UPO, interrupted line) and the optimized deformation of it that turns out to be a globally stable goal trajectory (SPO, solid line).



*Figure 2.* (a): Orbit in phase space of the difference vector  $\Delta \mathbf{z}$  between the UPO of the Lorenz system and the optimized stable goal trajectory (SPO). (b): Path of the corresponding vector control forces in the control signal space.

#### 4. Example

In this section, the control of a chaotic Lorenz system is demonstrated. The equations with vector additive control read

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) + u_1(t), & \dot{x}_2 &= rx_1 - x_2 - x_1x_3 + u_2(t), \\ \dot{x}_3 &= x_1x_2 - bx_3 + u_3(t), \end{aligned} \quad (7)$$

where time dependence of the forces is explicitly written. A given periodic goal trajectory  $\{\mathbf{z}(t), \dot{\mathbf{z}}(t)\}_{t \in [0, T]}$  yields control forces according to Eq. (4). Parameters of the Lorenz system are set to  $\sigma = 10$ ,  $r = 75$ , and  $b = 0.4$ . The resulting chaotic attractor is shown in Fig. 1(a). An embedded UPO which corresponds to  $\mathbf{z}_{UPO}$  in Eq. (5) is given in Fig. 1(b) by the interrupted line.

The deformation is defined by five Fourier modes ( $M=5$ ) which leads to a 34-dimensional search space for optimization;  $\gamma$  is set to 5,  $C$  to 100 in

Eq. (6). The best result of several optimization runs is shown in Fig. 1(b) by the solid line. It is a stable goal trajectory and therefore a stable periodic orbit (SPO) of the controlled system. The actual values of deformation parameters can be found in Tab. 1 together with additional data of the UPO and the SPO. The deformation lies in the range of some percent, and it is plotted in Fig. 2(a). The resulting forces, shown in Fig. 2(b), change the vector field of the chaotic system less than about 10%. This is an improvement of more than a magnitude if compared to goals in convergent regions [9].

Numerical tests indicated that the SPO is globally asymptotically stable; the basin is the whole phase space. However, transient times until control is established depend strongly on the initial state, and range from just a few up to a few hundred control periods. A typical behavior is presented in Fig. 3. After control is turned on, an intermittent transient appears. Finally, the system settles down on the desired goal orbit, which is maintained.

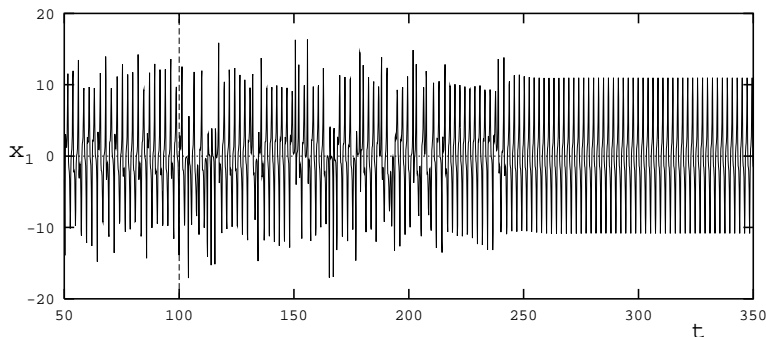


Figure 3. Controlled Lorenz system, first coordinate  $x_1(t)$ : Control is turned on at  $t_0 = 100$ , and the goal trajectory is reached after transients at about  $t = 250$  (after approximately 75 control periods).

## 5. Conclusion

It has been shown how open-loop entrainment control in the vicinity of unstable periodic orbits can be realized. The search for suitable goal trajectories has been formulated in terms of an optimization problem with respect to UPO deformations. Feasibility has been demonstrated in an example, where a chaotic Lorenz system has been successfully controlled to an optimized distortion of a UPO. The locations of such goal orbits are independent of convergent regions, and thus the required forces are small compared to hitherto used goal dynamics far from the chaotic attractor.

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TABLE 1. Coordinates  $z_{1,2,3}$  give a point of the original UPO of the Lorenz system,  $T$  its period. Maximum absolute values of the Floquet multipliers are given by  $\mu$  for the UPO and for the optimized deformation (SPO). Parameters of the SPO are the time transformation coefficient  $\eta$  and Fourier coefficients  $a_n^m$ ,  $b_n^m$ . Subscripts of numbers indicate a decimal shift, i.e.,  $1.234_{(-2)}$  stands for  $1.234 \times 10^{-2}$ .

$z_1 = 8.0$	$z_2 = 11.59618$	$z_3 = 68.87129$	$T = 2.153957$
$\mu_{UPO} = 2.555$	$a_1^0 = -9.794703_{-2}$	$a_1^1 = 2.303546_{-2}$	$b_1^1 = -1.946829_{-1}$
$\mu_{SPO} = 0.719$	$a_2^0 = -1.126072_{-1}$	$a_2^1 = -4.441229_{-2}$	$b_2^1 = -1.543789_{-1}$
$\eta = 1.001267$	$a_3^0 = 6.741204_{-1}$	$a_3^1 = -9.082832_{-2}$	$b_3^1 = 5.702340_{-2}$
$a_1^2 = 1.063004_{-1}$	$b_1^2 = 2.904178_{-5}$	$a_1^3 = 5.343983_{-2}$	$b_1^3 = -9.527600_{-3}$
$a_2^2 = 1.553288_{-1}$	$b_2^2 = -4.403841_{-2}$	$a_2^3 = 6.312158_{-2}$	$b_2^3 = -2.940390_{-2}$
$a_3^2 = -2.747584_{-1}$	$b_3^2 = -7.345061_{-2}$	$a_3^3 = 9.633147_{-2}$	$b_3^3 = 8.743705_{-2}$
$a_1^4 = 2.634346_{-2}$	$b_1^4 = 8.791274_{-2}$	$a_1^5 = 2.069052_{-3}$	$b_1^5 = 1.429860_{-2}$
$a_2^4 = 9.267735_{-2}$	$b_2^4 = 6.849559_{-2}$	$a_2^5 = 2.874568_{-2}$	$b_2^5 = 2.781748_{-2}$
$a_3^4 = 7.009472_{-2}$	$b_3^4 = 2.443292_{-2}$	$a_3^5 = -2.478029_{-2}$	$b_3^5 = 3.759263_{-2}$

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